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A Spectrum Generating Algebra for Meson Resonances

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Abstract

The algebra $SO(6,1)$ is considered as a unification of $SO(6)$, which is isomorphic to $SU(4) \supset SU(3)$, and the de Sitter algebra $SO(4,1)$. The latter replaces the Poincaré algebra as the algebra of the group of motions of physical space-time. A representation of $SO(6,1)$ is constructed, which, on restriction to $SU(3)$, decomposes into the direct sum of all $SU(3)$ representations, each occurring just once in the decomposition. The expectation values of the mass-squared operator, when evaluated in the octet, give accurate mass formulae for the octets of 1^- and 2^+ meson resonances.

1. *Introduction*

One of the problems of elementary particle physics is to understand the connection, if any, between the groups of motion L of physical space-time and the internal symmetry groups, such as $SU(3)$, the motivation being that some explanation may be found for the spin values and mass differences of particles in an $SU(3)$ multiplet. In attempting to find a group which contains both the internal and external symmetry groups as subgroups, the obvious approach to an economical solution is to find unifications (Flato & Sternheimer, 1966) $U(L, I)$ of L and I, so that the number of excess and therefore uninterpreted generators is kept to a minimum. Although the most plausible candidate for L, namely P the Poincaré group, has encountered serious difficulties (O'Raifeartaigh, 1965a, b, c; Jost, 1966; Segal, 1967), it cannot be discounted entirely since the 'no-go' theorems on the spectrum of the mass-squared operator $p_\mu p^\mu$ in an irreducible representation (I.R.) of the unifying algebra depend on the integrability of the algebra in a given representation space (Flato & Sternheimer, 1969).

The difficulties with the Poincaré group have become sufficiently acute to consider the possibility of replacing P by one of the de Sitter groups (both denoted by D) which are isomorphic to $SO(4,1)$ and $SO(3,2)$. Nothing can be done locally to distinguish between P and D as the group of motion of space-time, and both de Sitter groups tend to P in the local limit defined by the radius of the de Sitter universe approaching infinity. Furthermore, in this limit,

$$\lim_{R\to\infty} R^{-2} C_2 = p_\mu p^\mu$$

where C_2 is the second-order Casimir operator of D, which is not nilpotent (Roman, 1966). It is the nilpotency of P in its adjoint action on the unifying algebra which leads to the results of the theorems mentioned above. Theories using D have met with some success in finding a discrete mass formula for particles within a multiplet (Halbwachs, 1967). Vigier (1969) has considered finite dimensional representations of $SO(6,1)$, the algebra studied here, and has found a mass relation as a polynomial constraint in an over complete set of commuting operators.

Since there is no obvious choice for a unifying algebra, the problem of finding all possible unifications over the real field of D and a real simple internal symmetry algebra S was considered by Tait & Cornwell (1971). It was found that there are no unifications of D and $SU(3)$. The algebra $SO(6,1)$ is a unification of $SO(4,1)$ and $SO(6)$ which is isomorphic to $SU(4)$, which, in turn, contains $SU(3)$ as a subalgebra.

The concept of a unification, although useful as a minimality condition, is bound to lead to difficulties in interpretation because of the possibility of a non-zero intersection between D and S. Vigier (1969) has put forward a solution to this problem by assuming that there are two de Sitter groups D_1 and D_2 which are respectively the right and left translations of $SO(4,1)$. D_1 commutes with D_2, and their Casimir operators are equal. $D_1 \subset SO(6,1)$ is considered to be the 'internal' de Sitter algebra, while D_2, which commutes with $SO(6,1)$, is interpreted as the symmetry algebra of physical space-time. Another difficulty with unifications is that in choosing a complete set of commuting operators to construct an I.R., it is not always possible to have the desirable operators of D and S commuting with each other. Although it is inevitable for these reasons that algebras more general than unifications should be considered, the present study of unifications is necessary to indicate whether the results on the mass spectrum are reasonable, and to help in the formulation of new criteria for non-invariance algebras.

In Section 2 the $SO(6,1)$ algebra is considered, and bases are chosen for the relevant subalgebras. In Section 3, a unitary I.R. of $SO(6,1)$ is constructed, and on restriction to $SU(3)$ it is shown that the infinite dimensional representation contains every representation of $SU(3)$ just once. Section 4 contains the analysis of the mass-squared operator, showing that its diagonal matrix elements give accurate mass formulae for the octets of 1^- and 2^+ meson resonances.

2. *A Basis for the* $SO(6,1)$ *Algebra*

$SO(6,1)$ is the group of transformations which leave invariant the quadratic form,

$$-x_0^2 + \sum_{a=1}^{6} x_a^2$$

and the generators L_{AB} (A, $B = 0, 1, \ldots, 6$), where $L_{AB} = -L_{BA}$ satisfy the usual commutation relations for the algebra of a pseudo-orthogonal group, namely,

$$[L_{AB}, L_{CD}] = i(g_{AC}L_{BD} + g_{BD}L_{AC} - g_{AD}L_{BC} - g_{BC}L_{AD}) \qquad (2.1)$$

the metric tensor g_{KL} being defined by $g_{KL} = 0$, $K \neq L$ and

$$-g_{00} = g_{11} = \ldots = g_{66} = 1$$

$SO(6,1)$ contains $SO(6)$, which is locally isomorphic to $SU(4)$, as a subalgebra, and a basis for the latter is given below.

$$\begin{aligned} T_1 &= \tfrac{1}{2}(L_{23} - L_{15}) & T_2 &= \tfrac{1}{2}(L_{31} - L_{25}) \\ V_1 &= \tfrac{1}{2}(-L_{36} - L_{45}) & V_2 &= \tfrac{1}{2}(L_{34} + L_{56}) \\ U_1 &= \tfrac{1}{2}(-L_{24} + L_{16}) & U_2 &= \tfrac{1}{2}(-L_{14} - L_{26}) \\ F_1 &= \tfrac{1}{2}(L_{24} + L_{16}) & F_2 &= \tfrac{1}{2}(-L_{14} + L_{26}) \\ G_1 &= \tfrac{1}{2}(-L_{45} + L_{36}) & G_2 &= \tfrac{1}{2}(-L_{34} + L_{56}) \\ C_1 &= \tfrac{1}{2}(L_{23} + L_{15}) & C_2 &= \tfrac{1}{2}(L_{31} + L_{25}) \end{aligned}$$

$$\begin{aligned} T_3 &= \tfrac{1}{2}(L_{12} - L_{35}) \\ Y &= \tfrac{1}{3}(L_{12} + L_{35} - 2L_{46}) \\ Z &= \tfrac{1}{2}(L_{12} + L_{35} + L_{46}) \end{aligned}$$

The first twelve may be combined in the usual way to give the raising and lowering operators, $E_\alpha^\pm$ while the latter three provide the H_1, of a Cartan-Weyl basis. It can easily be seen that T, U, V, and Y, close on an $SU(3)$ subalgebra, with T_3 representing the third component of isospin, and Y the hypercharge. Furthermore Z, the 'charm' operator of $SU(4)$ theories, commutes with $SU(3)$.

The other subalgebra of interest, namely $SO(4,1)$ may be chosen to be

$$\{L_{\mu\nu};\ \mu, \nu = 0, 1, 2, 3, 5,\ \mu \neq \nu\}$$

Of course there are other bases for $SO(4,1)$ but the above choice is deliberate for the following reason. The spectrum of C_2 is to be examined, and since the isospin algebra lies entirely within $SO(4,1)$, the result is simplified by precluding mass splitting within an isomultiplet.

To be more specific, the following identification is made:

$$\left.\begin{aligned} \text{Momenta} \quad P_\mu &= L_{\mu 5} \\ \text{Space-time rotations } M_{\mu\nu} &= L_{\mu\nu} \end{aligned}\right\} \mu, \nu = 1, 2, 3, 0$$

The mass-squared operator is then

$$M^2 = -\frac{\hbar^2 c^2}{R^2}(P_\mu P^\mu + \tfrac{1}{2}M_{\mu\nu}M^{\mu\nu})$$

It may be seen that the difficulty mentioned in Section 1 has arisen, namely that the generators of $SO(4,1)$ do not commute with T_3, Y, or Z, implying that the internal quantum numbers are not relativistically invariant.

Vigier's argument must be invoked so that it is only the internal de Sitter transformation which can change the internal quantum numbers.

3.1. *I.R. of SO*(6, 1)

$SO(6,1)$ is one of the real forms of the complex simple Lie algebra B_3, which is of rank three and has twenty-one elements in a basis. Consequently a set of twelve commuting operators is necessary to specify completely any state within an I.R. In this case they are chosen to be the invariants of the algebras mentioned in the following chain:

$$SO(6,1) \supset SU(4) \supset SU(3) \otimes U(1)_Z \supset SU(2) \otimes U(1)_Y \supset U(1)_{T_3} \quad (3.1.1)$$

The representation to be constructed is akin to that used in the group theoretical description of the non-relativistic hydrogen atom, and the harmonic oscillator. In these problems the non-invariance groups are $SO(4,1)$ and $SU(3,1)$ respectively, whose representations are infinite towers of the symmetric tensor representations of the corresponding invariance groups, viz. $SO(4)$ and $SU(3)$.

The symmetric tensor, or most degenerate representations of $SO(6)$ are described by a single row of boxes in a Young diagram. They have been derived explicitly by Raczka *et al.* (1966), although the diagonalisation is not as shown in (3.1.1). The easiest way to construct the desired representation is to extend a result of Beg & Ruegg (1965) who have found the basis functions for all $SU(3)$ representations as harmonic functions on a five-dimensional surface of a sphere, S_5, embedded in a six-dimensional Euclidean space E_6.

The unit sphere S_5 may be parametrised in the following way:

$$\begin{aligned} Z_1 &= x_4 + ix_6 = r\,e^{i\phi_1}\cos\theta \\ Z_2 &= x_3 + ix_5 = r\,e^{i\phi_2}\sin\theta\cos\xi \\ Z_3 &= x_1 + ix_2 = r\,e^{i\phi_3}\sin\theta\sin\xi \end{aligned} \quad (3.1.2)$$

$$0 \leqslant \theta,\ \xi \leqslant \frac{\pi}{2}; \qquad 0 \leqslant \phi_k \leqslant 2\pi, \qquad k = 1,2,3; \qquad r = 1$$

It is easily seen that S_5, i.e.

$$Z_k Z_k^* = x_a x_a = r^2 = 1$$

is invariant under the action of $SO(6)$ as well as $SU(3)$. The invariant metric G_{ij} on S_5 is then given by

$$\begin{aligned} ds^2 &= |dz_1|^2 + |dz_2|^2 + |dz_3|^2 \\ &= d\theta^2 + \cos^2\theta\, d\phi_1{}^2 + \sin^2\theta(d\xi^2 + \cos^2\xi\, d\phi_2{}^2 + \sin^2\xi\, d\phi_3{}^2) \end{aligned}$$

The basis functions for the I.R. of $SO(6)$ are the eigenfunctions of the Laplace–Beltrami operator on S_5,

$$\Delta_5 = G^{-1/2}\frac{\partial}{\partial\eta^i}\left(G^{1/2}G^{ij}\frac{\partial}{\partial\eta^j}\right); \qquad i,j = 1,2,3,4,5$$

where $G = \det(G_{ij})$, $G^{ij} = (G^{-1})_{ij}$, and $\{\eta^i\} = \theta,\ \xi,\ \phi_k$.

The eigenvalues of Δ_5 are $-n(n+4)$, n being a positive integer, so that the equation remaining to be solved is

$$\Delta_5 Y_n + n(n+4) Y_n = 0$$

which is identical to the equation for the basis functions found by Raczka *et al.* (1966). However, the method of solution is different, in this case, to bring out the diagonalisation (3.1.1). The solution is found to be the product of two Wigner d-functions weighted by $\operatorname{cosec}\theta$, i.e.

$$Y_n = \Psi^{p,q}_{T,T_3,Y} = \operatorname{cosec}\theta\, d(2\theta)^{\frac{1}{2}(p+q+1)}_{\frac{1}{6}(p-q-3Y+6T+3),\frac{1}{6}(p-q-3Y-6T-3)}$$
$$.\, d(2\xi)^{T}_{\frac{1}{3}(p-q)+\frac{1}{2}Y,T_3} \,.\exp\tfrac{1}{3}i(p-q)(\phi_1+\phi_2+\phi_3)$$
$$.\exp iT_3(\phi_2-\phi_3)\,.\exp\tfrac{1}{2}iY(-2\phi_1+\phi_2+\phi_3)$$

where $p+q=n$, and $\frac{1}{2}(p-q)$ is the eigenvalue of the charm operator Z. (p,q) is the pair of positive integers which specify an I.R. of $SU(3)$. For fixed n, the set of functions Y_n is irreducible under $SO(6)$. The idea now is to construct a representation of $SO(6,1)$ on the space of functions

$$\bigoplus_{n=0}^{\infty} Y_n \tag{3.1.3}$$

but before doing this, some properties of the $SO(6)$ representation will be discussed.

3.2. *The Restriction of the SO(6) Representation to SU(3)*

The functions Y_n (n fixed) may be expressed as

$$Y_n = \bigoplus_{p,q;p+q=n} \psi^{p,q}$$

where for fixed p and q, $\psi^{p,q}$ is irreducible under $SU(3)$. In the chain (3.1.1) there is only one additive quantum number which can distinguish between different I.R.'s of $SU(3)$ contained in an I.R. of $SO(6)$, namely the eigenvalue Z of the charm operator which goes in integer steps from $-n/2$ to $+n/2$. Since $n=p+q$ is fixed, when a value for $Z=\frac{1}{2}(p-q)$ is chosen, both p and q are determined. It follows that there is only one representation of $SU(3)$ corresponding to each value of $\frac{1}{2}(p-q)$. Clearly, as n takes all positive integral values (as it will do in the $SO(6,1)$ representation) all the representations of $SU(3)$ occur just once.

It can easily be shown that the n-symmetric tensor representations of $SO(6)$ correspond to those representations of $SU(4)$, described by a Young diagram consisting of two rows of n boxes. Hence it is possible to use well-known results (Amati *et al.*, 1964) on $SU(4)$ to give the dimensions of the representations, and their decomposition on restriction to $SU(3)$. These are summarised in Table 1 for a few values of n.

TABLE 1. The dimensions of the symmetric tensor representations of $SO(6)$ and their decomposition w.r.t. $SU(3)$.

n	Dim. of $SO(6)$ rep.	Decomposition $SU(3)$
0	1	1
1	6	$3+\overline{3}$
2	20	$6+8+\overline{6}$
3	50	$10+15'+\overline{15}'+\overline{10}$
4	105	$15+24+27+\overline{24}+\overline{15}$

3.3 *Generators for the Hermitian I.R. of* $SO(6,1)$

To realise a representation of $SO(6,1)$ on the space of functions (3.1.3), the action of the generators L_{AB} must be specified. The $SO(6)$ subalgebra L_{ab} $(a,b=1,2,\ldots,6)$ may be represented by differential operators of the form

$$L_{ab}=i\left(x_b\frac{\partial}{\partial x_a}-x_a\frac{\partial}{\partial x_b}\right)$$

where the x_a are defined in (3.1.2); and by using the spherical parametrisation (3.1.2) with $0\leqslant r\leqslant\infty$ the L_{ab} may be recast in the form

$$L_{ab}=\sum_{i=1}^{5}f_i(\eta_j)\frac{\partial}{\partial\eta_i} \tag{3.3.1}$$

(3.3.1) is independent of r and $\partial/\partial r$ as expected, since the Y_n are independent of r.

The specification of the generators L_{0a} is more involved, and is accomplished by using the methods of Budini (1966) or Bander & Itzykson (1966). Consider the stereographic projection from E_6 to the six-dimensional surface of a hyperboloid H_6, defined by

$$y_a=\frac{2cx_a}{c^2-x^2},\qquad y_0=\frac{c^2+x^2}{c^2-x^2} \tag{3.3.2}$$

where c is a constant. The seven-vector y lies on the hyperboloid, since

$$-y_0{}^2+y_a y_a=1 \tag{3.3.3}$$

The group leaving (3.3.3) invariant is $SO(6,1)$ with generators represented by

$$L_{ab}=-iy_a\frac{\partial}{\partial y_b}+iy_b\frac{\partial}{\partial y_a}$$

$$L_{0a}=iy_0\frac{\partial}{\partial y_a}+iy_a\frac{\partial}{\partial y_0} \tag{3.3.4a}$$

Using (3.3.2), the latter give

$$L_{ab} = -ix_a \frac{\partial}{\partial x_b} + ix_b \frac{\partial}{\partial x_a}$$

$$L_{0a} = -i\frac{(c^2+x^2)}{2c}\frac{\partial}{\partial x_a} + i\frac{x_a}{c}\sum_{d=1}^{6} x_d \frac{\partial}{\partial x_d} \tag{3.3.4b}$$

An inner product is defined by

$$(Y_m, Y_n) = \int d\mu . Y_m^* \, Y_n$$

where $d\mu$ is the surface element on S_5, i.e.

$$d\mu = \cos\theta \sin^3\theta \sin\xi \cos\xi \, d\theta \, d\xi \, d\phi_1 \, d\phi_2 \, d\phi_3$$

However (3.3.4a) and (3.3.4b) are hermitian not with respect to this measure, but with respect to

$$dy_0 \, dy_1 \ldots dy_6$$

In order to take account of the different measures, operators V_{AB} are defined by

$$V_{AB} = UL_{AB} U^{-1}$$

where

$$U = \left(\frac{2c}{c^2 - x^2}\right)^{5/2}$$

This gives

$$V_{ab} = -ix_a \frac{\partial}{\partial x_b} + ix_b \frac{\partial}{\partial x_a} \tag{3.3.5}$$

$$V_{0a} = L_{0a} + \tfrac{5}{2} i \frac{x_a}{c}$$

$$= -i\frac{(c^2+x^2)}{2c}\frac{\partial}{\partial x_a} + i\frac{x_a}{c}\sum_d x_d \frac{\partial}{\partial x_d} + \tfrac{5}{2} i \frac{x_a}{c} \tag{3.3.6}$$

The choice $x^2 = c^2 = 1$ ensures that the representation is based on the spherical harmonics on S_5, since this removes the $\partial/\partial r$ dependence of V_{0a}, and the V_{0a} become

$$V_{0a} = -i\frac{\partial}{\partial x_a} + ix_a \sum_d x_d \frac{\partial}{\partial x_d} + \tfrac{5}{2} i x_a \tag{3.3.7}$$

(3.3.5) and (3.3.7) are now hermitian with respect to the measure $d\mu$. The operators V_{01}, V_{02}, V_{03} and V_{05} are given explicitly since they will be used in the mass-squared operator in the next section. They are

$$V_{01} = -i\,\mathrm{c}\phi_3\,\mathrm{s}\xi\,\mathrm{c}\theta\frac{\partial}{\partial\theta} - i\frac{\mathrm{c}\phi_3\,\mathrm{c}\xi}{\mathrm{s}\theta}\frac{\partial}{\partial\xi} + i\frac{\mathrm{s}\phi_3}{\mathrm{s}\theta\,\mathrm{s}\xi}\frac{\partial}{\partial\phi_3} + \tfrac{5}{2}i\,\mathrm{c}\phi_3\,\mathrm{s}\theta\,\mathrm{s}\xi$$

$$V_{02} = -i\,\mathrm{s}\phi_3\,\mathrm{s}\xi\,\mathrm{c}\theta\frac{\partial}{\partial\theta} - i\frac{\mathrm{s}\phi_3\,\mathrm{c}\xi}{\mathrm{s}\theta}\frac{\partial}{\partial\xi} - i\frac{\mathrm{c}\phi_3}{\mathrm{s}\theta\,\mathrm{s}\xi}\frac{\partial}{\partial\phi_3} + \tfrac{5}{2}i\,\mathrm{s}\phi_3\,\mathrm{s}\theta\,\mathrm{s}\xi$$

$$V_{03} = -i\,\mathrm{c}\phi_2\,\mathrm{c}\xi\,\mathrm{c}\theta\frac{\partial}{\partial\theta} + i\frac{\mathrm{c}\phi_2\,\mathrm{s}\xi}{\mathrm{s}\theta}\frac{\partial}{\partial\xi} + i\frac{\mathrm{s}\phi_2}{\mathrm{s}\theta\,\mathrm{c}\xi}\frac{\partial}{\partial\phi_2} + \tfrac{5}{2}i\,\mathrm{c}\phi_2\,\mathrm{s}\theta\,\mathrm{c}\xi$$

$$V_{05} = -i\,\mathrm{s}\phi_2\,\mathrm{c}\xi\,\mathrm{c}\theta\frac{\partial}{\partial\theta} + i\frac{\mathrm{s}\phi_2\,\mathrm{s}\xi}{\mathrm{s}\theta}\frac{\partial}{\partial\xi} - i\frac{\mathrm{c}\phi_2}{\mathrm{s}\theta\,\mathrm{c}\xi}\frac{\partial}{\partial\phi_2} + \tfrac{5}{2}i\,\mathrm{s}\phi_2\,\mathrm{s}\theta\,\mathrm{c}\xi \quad (3.3.8)$$

where $\mathrm{c} \equiv \cos$, $\mathrm{s} \equiv \sin$.

4.1. *The Mass-Squared Operator*

From the C.R. (2.1) it can be seen that the mass-squared operator M^2 commutes with T^2, T_3, Y and Z, thereby causing no transitions between the states of an $SU(3)$ or even $SU(4)$ multiplet. However M^2 does not commute with the Casimir operators of $SU(3)$ so that it is possible to have mixing between different multiplets, and, from the general form of the operators V_{0a} (3.3.7), it can be anticipated that the general non-zero matrix elements of M^2 are of the form $\langle p,q|M^2|p,q\rangle$ and $\langle p\pm 1, q\pm 1|M^2|p,q\rangle$, where $|p,q\rangle$ is the spherical harmonic $\psi^{p,q}$.

At this point it must be noted that the only unknown parameter in M^2 is the multiplicative factor $\lambda^2 = \hbar^2 c^2/R^2$ which is the natural unit of (mass)^2 in a de Sitter space-time. This is obviously very small since R is so large, so that for a sensible definition of mass, it is necessary to scale up M^2 by multiplying by a factor R^2/r^2, where r is of the order of a Compton wavelength of an elementary particle. The mass-squared operator then becomes

$$M^2 = \mu^2(V_{01}^2 + V_{02}^2 + V_{03}^2 + V_{05}^2 - V_{12}^2 - V_{23}^2 - V_{31}^2 - V_{15}^2 - V_{25}^2 - V_{35}^2)$$

where $\mu^2 = \hbar^2 c^2/r^2$ and must be determined by comparison with experiment. Alternatively, this may be written

$$M^2 = \mu^2(V_{01}^2 + V_{02}^2 + V_{03}^2 + V_{05}^2 - 2T^2 - 2C^2)$$

where $C^2 = C_1{}^2 + C_2{}^2 + C_3{}^2$ and $C_3 = \tfrac{1}{2}(L_{12} + L_{35})$. C^2 is always diagonal in the representation being considered, and in fact it may be shown that it has the same eigenvalues as T^2, i.e.

$$C^2\psi^{p,q} = T(T+1)\psi^{p,q}$$

Now consider the other part of M^2. Using (3.3.8),

$$V^2 \equiv V_{01}^2 + V_{02}^2 + V_{03}^2 + V_{05}^2$$
$$= -c^2\theta \frac{\partial^2}{\partial\theta^2} - \frac{1}{s^2\theta}\frac{\partial^2}{\partial\xi^2} - \frac{1}{s^2\theta\, s^2\xi}\frac{\partial^2}{\partial\phi_3{}^2} - \frac{1}{s^2\theta\, c^2\xi}\frac{\partial^2}{\partial\phi_2{}^2}$$
$$+ 3\frac{c\theta}{s\theta}(2s^2\theta - 1)\frac{\partial}{\partial\theta} + \frac{1}{s^2\theta}[\tan\xi - \cot\xi]\frac{\partial}{\partial\xi}$$
$$- \tfrac{25}{4}s^2\theta + \tfrac{5}{2}c^2\theta + \tfrac{15}{2}$$

The ξ dependence of this operator may be eliminated by noting that,

$$4T^2 = -\frac{\partial^2}{\partial\xi^2} + (\tan\xi - \cot\xi)\frac{\partial}{\partial\xi} - \frac{1}{c^2\xi}\frac{\partial^2}{\partial\phi_2{}^2} - \frac{1}{s^2\xi}\frac{\partial^2}{\partial\phi_3{}^2}$$

so that

$$V^2 = -c^2\theta\frac{\partial^2}{\partial\theta^2} + 3\frac{c\theta}{s\theta}(2s^2\theta - 1)\frac{\partial}{\partial\theta} + \frac{4T^2}{s^2\theta} - \tfrac{25}{4}s^2\theta + \tfrac{5}{2}c^2\theta + \tfrac{15}{2}$$

This expression can be simplified even further using the fact that I_2, the second-order invariant of $SU(3)$, with eigenvalues

$$\tfrac{1}{9}(p^2 + q^2 + pq + 3p + 3q)$$

is given by

$$12I_2 = -\frac{\partial^2}{\partial\theta^2} - (3\cot\theta - \tan\theta)\frac{\partial}{\partial\theta} + \frac{4T^2}{s^2\theta} + \frac{Y^2}{c^2\theta} + \tfrac{1}{3}(p-q)^2$$

Using this to define $\partial^2/\partial\theta^2$, it follows that

$$M^2 = \mu^2\Big[6I_2 + \tfrac{45}{8} + (6I_2 + \tfrac{35}{8})c\,2\theta + 2s\theta\, c\theta\frac{\partial}{\partial\theta} - \tfrac{1}{3}(p-q)^2c^2\theta$$
$$+ 2T^2 - 2c^2 - Y^2\Big]$$

To define the action of M^2 on $\psi^{p,q}$ it is necessary to consider only

$$M^2 \operatorname{cosec}\theta\, d\underset{\frac{1}{2}(p-q-3Y+6T+3),\frac{1}{2}(p-q-3Y-6T-3)}{\overset{\frac{1}{2}(p+q+1)}{(2\theta)}}$$

which can be evaluated using standard manipulations with d-functions. These are quoted in the appendix to this paper. At this point the argument will be limited to the case $p = q$ since it will be seen that this formalism applies to meson multiplets. Now, using equations (A.1), (A.2), (A.3), and the normalised spherical harmonics on S_5 i.e.

$$2[(2T+1)(p+q+2)]^{1/2}\psi^{p,q}_{T,T_3,Y}$$

the diagonal matrix elements of M^2 are found to be

$$\langle p|M^2|p\rangle = \mu^2\Big\{\tfrac{1}{2}(2p+1)(2p+3) + \frac{1}{8(2p+1)(2p+3)} + \tfrac{21}{8} - Y^2\Big[\tfrac{1}{2} + \frac{1}{8(2p+1)(2p+3)}\Big] + T(T+1)\Big[\frac{1}{2(2p+1)(2p+3)} - 2\Big]\Big\} \qquad (4.1.1)$$

For a physically interesting multiplet, e.g. the octet of $SU(3)$, $p=1$ and (4.1.1) reduces to

$$\langle M^2\rangle = \tfrac{304}{30}\mu^2 - \tfrac{59}{30}\mu^2[T(T+1) + \tfrac{61}{236}Y^2] \qquad (4.1.2)$$

Note that $\frac{61}{236}$ is very close to $\frac{1}{4}$, and hence there is a similarity between (4.1.2) and the Gell–Mann–Okubo mass formula for mesons,

$$\langle M^2\rangle = M_0{}^2 - b^2[T(T+1) - \tfrac{1}{4}Y^2] \qquad (4.1.3)$$

where $M_0{}^2$ and b^2 are independent parameters, which means that (4.1.2) has more predictive power than (4.1.3). The latter is known to fit the pseudoscalar mesons very well, but gives poor results when applied to the octet of vector mesons. This is thought to be due to $\omega-\phi$ mixing. However (4.1.2) does fit the vector mesons with the *physical* ϕ in the octet. With $\mu^2 = 0{\cdot}09914$ GeV^2,

$$\rho^2 = 0{\cdot}615\ (0{\cdot}587)\ \text{GeV}^2$$
$$K^{*2} = 0{\cdot}808\ (0{\cdot}797)\ \text{GeV}^2$$
$$\phi^2 = 1{\cdot}004\ (1{\cdot}039)\ \text{GeV}^2$$

showing good agreement with the experimental values (Rosenfeld, 1969) in brackets. It is a little unfortunate that the singlet ($p=0$) is not the ω-particle, since $M^2 = 0{\cdot}414$ GeV^2 for the above value of μ^2.

4.2. *The Eigenvalues of M^2*

The only non-zero off-diagonal matrix elements of M^2 are found to be (for $p=q$)

$$\langle p-1|M^2|p\rangle = \frac{\mu^2(2p^2+2p+\frac{3}{8})}{4(p+1)(2p+1)}\cdot\left(\frac{p+1}{p}\right)^{1/2} .[(2p-Y-2T)(2p+Y-2T)(2p+Y+2T+2)(2p-Y+2T+2)]^{1/2} \qquad (4.2.1)$$

and

$$\langle p+1|M^2|p\rangle = \frac{\mu^2(2p^2+6p+\frac{35}{8})}{4(p+1)(2p+3)}\cdot\left(\frac{p+1}{p+2}\right)^{1/2} .[(2p-Y-2T+2)(2p+Y-2T+2)(2p+Y+2T+4)(2p-Y-2T+4)]^{1/2} \qquad (4.2.2)$$

the hermiticity of M^2 may be checked by putting $p = p' - 1$ in the latter to show that $\langle p-1|M^2|p\rangle = \langle p|M^2|p-1\rangle$. In view of the good agreement with experiment found in the previous section, it might be expected that the off-diagonal elements are small compared to those on the diagonal. This is not the case, and both (4.2.1) and (4.2.2) are monotonically increasing functions of p, $\sim p^2$ for large p, just like $\langle M^2\rangle$. The M^2 matrix may be diagonalised, by approximating the infinite dimensional matrix by a finite dimensional one of order $(p+1)$, but because the off-diagonal elements are so large, the convergence of the eigenvalues for increasing p is slow. At about $p = 70$, the variation with p is slow enough for the asymptotes $m^2(T, Y)$ to be found by means of the parametrisation

$$m^2(T, Y) + \frac{a}{p}(T, Y) + \frac{b}{p^2}(T, Y) \tag{4.2.3}$$

Table 2 shows the variation in the eigenvalues of interest with the size of the matrix. (The factor μ^2 has been suppressed).

TABLE 2. The eigenvalues of particle masses varying with the dimension N of the approximating matrix.

N	Singlet S^2	ϕ^2	ρ^2	K^{*2}
10	2·796	4·869	3·176	3·654
20	2·655	4·088	2·893	3·198
30	2·595	3·785	2·781	3·019
40	2·560	3·613	2·717	2·918
50	2·539	3·501	2·675	2·853
60	2·521	3·418	2·647	2·897
70	2·508	3·356	2·627	2·771

The ratios $\phi^2/\rho^2, \phi^2/K^{*2}, K^{*2}/\rho^2$ and ϕ^2/s^2 also form asymptotic sequences with increasing N. Using a formula of the form (4.2.3), it was found that $\phi^2/K^{*2} = 1{\cdot}155$, $K^{*2}/\rho^2 = 1{\cdot}024$. These imply that $\phi^2/\rho^2 = 1{\cdot}18$. An independent check on this quantity gave $\phi^2/\rho^2 = 1{\cdot}17$. Also, $\phi^2/s^2 = 1{\cdot}21$.

However, these ratios do not fit the vector mesons, or any other known octet of mesons. In addition to this result, it may be argued that even the concept of an $SU(3)$ multiplet has disappeared in the diagonalisation process. The only consolation is that a definite splitting in the mass-squared eigenvalues, varying with T and Y, has been established. The fact that the expectation values give agreement with experiment may be indicating that an algebra which is larger than $SO(6,1)$, and which avoids the intersection of $SO(4,1)$ and $SU(3)$, is necessary.

4.3. *The Octet of* 2^+ *Mesons*

It can easily be checked that the C.R. of the operators representing $SO(6,1)$ are unaffected by the addition of a term proportional to x_a to V_{0a}, i.e.

$$V_{0a} \to V_{0a} + \gamma x_a \tag{4.2.4}$$

and the representation remains hermitian if γ is real. The evaluation of $\langle M^2 \rangle$ with the substitution (4.2.4) gives a two parameter (μ^2, γ) mass formula, which when restricted to the octet, fits the 2^+ mesons (A_2, K_N, f'). The mass-squared expectation values are

$$\begin{aligned} A_2{}^2 &= \mu^2(\tfrac{31}{5} + \tfrac{4}{5}\gamma^2) \\ K_N{}^2 &= \mu^2(\tfrac{163}{20} + \tfrac{3}{5}\gamma^2) \\ f'^2 &= \mu^2(\tfrac{152}{15} + \tfrac{8}{15}\gamma^2) \end{aligned} \tag{4.2.5}$$

which lead to the sum rule

$$36K_N{}^2 = 21f'^2 + 13A_2{}^2 \tag{4.2.6}$$

with left-hand side $= 72{\cdot}36$ GeV^2 and right-hand side $= 70{\cdot}19$ GeV^2. γ^2 may be estimated from the ratio of any two equations in (4.2.5), giving $\gamma^2 \sim 3{\cdot}0$. μ^2 then turns out to be twice the value found for the vector mesons, i.e. $\mu^2 = 2 \times 0{\cdot}09914$ GeV^2.

For the singlet S,

$$S^2 = \mu^2(\tfrac{25}{6} + \tfrac{2}{3}\gamma^2)$$

and the above values for μ^2 and γ^2 give $S^2 = 1{\cdot}22$ GeV^2. There appears to be no 2^+ object recorded at this mass value but (this may be just a coincidence) $S^2 \doteqdot 2\omega^2$ showing that if the original estimate of μ^2 is adopted, the mass of the ω-particle is predicted.

An alternative transformation which leaves the representation unchanged is

$$V_{0a} \to \left(V_{0a} + it\frac{\partial}{\partial x_a}\right)(1-t)^{-1/2} \tag{4.2.7}$$

(t real), the factor $(1-t)^{-1/2}$ being necessary because

$$\left[V_{0a} + it\frac{\partial}{\partial x_a}, V_{0b} + it\frac{\partial}{\partial x_b}\right] = -i(1-t)\, V_{ab}$$

With the substitution (4.2.7), the expectation values of M^2 in the octet become

$$\begin{aligned} A_2{}^2 &= \frac{\mu^2(31 - 130t)}{5(1-t)} \\ K_N{}^2 &= \frac{\mu^2(163 - 460t)}{20(1-t)} \\ f'^2 &= \frac{\mu^2(152 - 270t)}{15(1-t)} \end{aligned} \tag{4.2.8}$$

and by eliminating μ^2 and t, the exact sum rule is

$$4556K_N{}^2 = 2079f'^2 + 2591A_2{}^2$$

which will be approximated by

$$4{\cdot}56K_N{}^2 = 2{\cdot}08f'^2 + 2{\cdot}59A_2{}^2 \qquad (4.2.9)$$

This is in excellent agreement with experiment, since left-hand side = 4·76 GeV^2 and right-hand side = 4·76 GeV^2.

It is worth noting that both sum rules (4.2.6) and (4.2.9) satisfy the octet of vector mesons, with the predictions $28{\cdot}7 \doteqdot 29{\cdot}4$ GeV^2 and $2{\cdot}16 \doteqdot 2{\cdot}11$ GeV^2 respectively, when the particle labels are interchanged. From (4.2.8), $t = -0{\cdot}115$ and as before $\mu^2 = 2 \times 0{\cdot}09914$ GeV^2. However, with these values, the singlet state does not correspond to any known particle:

$$S^2 = \frac{\mu^2(25-60t)}{6(1-t)} = 4{\cdot}77\mu^2$$

Discussion

It is quite surprising that the algebra $SO(6,1)$, chosen for no other reason than that it is the smallest unification of $SO(4,1)$ and an internal symmetry algebra containing $SU(3)$, should lead to such accurate mass formulae. Also there is a conceptual difficulty in understanding how a small curvature in space-time can be important in the region of an elementary particle. However, this would not be a difficulty if the no-go theorems on the spectrum are simply mathematical results with no reference to physics.

An alternative interpretation of this work would be to imagine that space-time has a very large curvature in the region occupied by an elementary particle. Such an idea has been suggested and examined by Barut & Böhm (1965) and Böhm (1966) who were able to find a formula relating the masses and spins of particles and their resonances. If this approach is adopted, the scaling-up procedure described in Section 4.1 becomes unnecessary, and the prediction for the radius of an elementary particle is $R \sim 10^{-13}$ cms.

It must be noted that there is no *a priori* way of telling which particular octets of mesons are being described by the representation used here, since the analogy of the spin operator for the de Sitter algebra is zero. Perhaps it may be possible to include spin in a more general representation, and also find a mass formula for spin 0 mesons. In a forthcoming paper, a tentative extension of this scheme to include baryon resonances will be discussed.

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APPENDIX

The standard results on d-functions needed to evaluate the matrix elements of M^2 are:

$$\cos 2\theta\, d^l_{j,m}(2\theta) = \frac{[(l+m)(l-m)(l+j)(l-j)]^{1/2}}{l(2l+1)} d^{l-1}_{j,m}(2\theta) + \frac{mj}{l(l+1)} d^l_{jm}(2\theta) + \frac{[(l+m+1)(l-m+1)(l+j+1)(l-j+1)]^{1/2}}{(2l+1)(l+1)} d^{l+1}_{j,m}(2\theta) \quad \text{(A.1)}$$

$$\sin 2\theta\, d^l_{j,m}(2\theta) = \frac{i[(l+m)(l+m+1)(l+j+1)(l-j+1)]^{1/2}}{(l+1)(2l+1)} d^{l+1}_{j,m}(2\theta) - \frac{ij[(l+m)(l-m+1)]^{1/2}}{l(l+1)} d^l_{j,m}(2\theta) - \frac{i[(l-m)(l-m+1)(l+j)(l-j)]^{1/2}}{l(2l+1)} d^{l-1}_{j,m}(2\theta) \quad \text{(A.2)}$$

$$2\sin\theta\cos\theta \frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta} d^l_{j,m}(2\theta)\right] = \frac{1}{\sin\theta}[(2j-1)\, d^l_{j,m}(2\theta) - (2m+1)\, d^l_{j,m}(2\theta)\cos 2\theta - 2i\alpha_m \sin 2\theta\, d^l_{j,m-1}(2\theta)] \quad \text{(A.3)}$$

where

$$\alpha_m = [(l+m)(l-m+1)]^{1/2}$$